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Singular hypersurfaces in the Brans–Dicke theory of gravity

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Abstract. Junction conditions are formulated in an invariant manner for the jumps in the gravitational and scalar fields across a singular time-like hypersurface in the Brans–Dicke theory of gravity. The equations of motion for singular hypersurfaces are also derived, and an exact solution of the Brans–Dicke field equations is presented which represents a static spherical shell of perfect fluid.

1. Introduction

Gravitational junction conditions for singular hypersurfaces in general relativity were originally formulated by Lanczos (1922, 1924). A formulation in terms of extrinsic and intrinsic curvature was developed by Darmois (1927), Misner and Sharp (1964), and Israel (1966, 1967a) who expressed the junction conditions in an invariant form. The charged case was then treated by Kuchař (1968).

We consider here junction conditions for singular time-like hypersurfaces in the Brans–Dicke theory of gravity, but restrict our attention to uncharged hypersurfaces in vacuum. Recent observations limit the Dicke coupling constant ω to values greater than $\omega = 500$ (Reasenberg *et al* 1979), so that Brans–Dicke theory today is very much the 'large-ω version' discussed by Thorne and Dykla (1971). When ω is large enough for the scalar field to be treated as a perturbation, Thorne and Dykla proved (using Price's (1972) analysis) that, when a non-rotating star collapses, all the scalar field is radiated away leaving a Schwarzschild black hole. This, however, does not answer the question of what happens to a collapsing star with an arbitrarily strong scalar field. Is all the scalar field radiated away in this case too? It probably is, but no successful numerical calculations of such a collapse have been performed, and the answer, in spite of the above observational limits on ω, is still of theoretical interest. The same applies to the junction conditions themselves and the study of singular hypersurfaces in Brans–Dicke theory. One way of seeing if all the scalar field is radiated away during collapse would be to study numerically the collapse of a neutron star in Brans–Dicke theory, but a previous attempt at this problem (Matsuda and Nariai 1973) proved inconclusive because of numerical difficulties encountered during the latter stages of collapse.

In general relativity the study of the dynamics of collapsing spherical shells by Israel (1967b), De la Cruz and Israel (1967), Papapetrou and Hamoui (1968) and Chase (1972) led to important physical insights about the collapse process. This was

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accomplished with a minimum of effort because shells are simpler to deal with than whole stars. This is particularly true for spherical shells of dust where the equation of motion can be solved analytically (Israel 1967b). In the Brans–Dicke theory spherical shells of dust should again provide the simplest situation in which to analyse collapse, and consequently one of the aims of the present work is to derive the equations of motion of singular hypersurfaces in Brans–Dicke theory. The emission of scalar gravitational radiation means that the study of collapsing shells of dust, even in the case of spherical symmetry, is far from trivial, and an analytic solution is probably not possible. Because of the added complexity of the scalar radiation, we do not consider the problem of collapse in this paper, but content ourselves with deriving the equations of motion.

In § 2 singular hypersurfaces are defined and the junction conditions are derived for the scalar and gravitational fields. The equations of motion of singular hypersurfaces are derived in § 3, and finally a simple application of the formulae of § 2 appears in § 4 where a solution is discussed that represents a static spherical shell of perfect fluid.

2. Singular hypersurfaces in the Brans–Dicke theory

2.1. *Definition of singular hypersurfaces*

A time-like singular hypersurface Σ in vacuum is the history of an infinitely thin layer of matter, which may be regarded as the limit as $\varepsilon \to 0$ of a layer of finite thickness 2ε. For such a finite-thickness layer in vacuum with stress–energy tensor ${}^{(4)}T_{\alpha\beta}$ the surface stress–energy tensor is defined as

$$ {}^{(4)}S_{\alpha\beta} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} {}^{(4)}T_{\alpha\beta}\, \mathrm{d}n. \tag{1} $$

Here n is proper distance measured perpendicular to Σ, and Σ is at $n = 0$ (see e.g. Israel 1966, Misner *et al* 1973). We may alternatively regard ${}^{(4)}T_{\alpha\beta}$ as developing a delta function in the limit $\varepsilon \to 0$, so that formally

$$ {}^{(4)}T_{\alpha\beta} \to \delta(n)\,{}^{(4)}S_{\alpha\beta} \tag{2} $$

which is consistent with (1) (Papapetrou and Hamoui 1968).

The three-dimensional surface stress–energy tensor S_{ab} (where latin indices label the intrinsic coordinates ξ^a) is now defined to be the projection of $S_{\alpha\beta}$ onto Σ in the manner described by Kuchař (1968). The only non-vanishing tetrad component of $S_{\alpha\beta}$ is

$$ S_{ab} = S_{\alpha\beta}\, e_{(a)}{}^{\alpha} e_{(b)}{}^{\beta} $$

where $e_{(a)}{}^{\alpha}$ is a time-like tangent vector. The trace S of $S_{\alpha\beta}$ can be defined by $S = g^{\alpha\beta} S_{\alpha\beta}$ or by

$$ S = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} {}^{(4)}T\, \mathrm{d}n. \tag{3} $$

2.2. *Brans–Dicke field equations and junction conditions for the scalar field*

The field equations of Brans and Dicke (1961) are

$$ G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = -8\pi\phi^{-1} T_{\alpha\beta} - \Phi_{\alpha\beta}. \tag{4} $$

Here

$$\Phi_{\alpha\beta} = \omega\phi^{-2}(\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}g_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma}) + \phi^{-1}(\nabla_\beta\phi_{,\alpha} - g_{\alpha\beta}\Box\phi) \tag{5}$$

and the source of the scalar field ϕ is the trace of the stress–energy tensor through the wave equation

$$\Box\phi = \frac{8\pi}{2\omega+3}\,{}^{(4)}T \tag{6}$$

where

$$\Box\phi = g^{\alpha\beta}\nabla_\beta\phi_{,\alpha} = \nabla^\alpha\nabla_\alpha\phi$$

and ∇_α is the four-dimensional covariant derivative.

We consider in this section how ϕ and its normal derivative change across Σ. The scalar field itself must be continuous across Σ, otherwise its value in Σ cannot be defined uniquely. In addition, if ϕ suffered, for example, a step discontinuity across Σ, its derivative in the direction normal to Σ would contain a delta function at Σ and the right-hand side of equation (6) would have to contain the derivative of a delta function. This is not the case, however, because from equation (2) ${}^{(4)}T$ contains only a delta function, not the derivative of a delta function. We thus take

$$[\phi] = 0 \tag{7}$$

as the junction condition for ϕ where $[\phi]$ denotes the jump in ϕ across Σ.

The normal derivative of ϕ, defined by

$$\phi_{,n} = \phi_{,\alpha}n^\alpha$$

where n^α is the space-like normal vector to Σ, does suffer a discontinuity across Σ, as follows from the wave equation (6). To calculate this discontinuity start by evaluating the tetrad components of $\nabla_\beta\phi_{,\alpha}$ which appear in $\Box\phi$. These are obtained by decomposing $\phi_{,\alpha}$ as

$$\phi_{,\alpha} = \phi_{,n}n_\alpha + \phi_{,a}e^{(a)}{}_\alpha \tag{8}$$

(cf Kuchař 1968) where

$$\phi_{,a} = \phi_{,\alpha}e_{(a)}{}^\alpha.$$

Next operate on (8) with ∇_β to obtain

$$\nabla_\beta\phi_{,\alpha} = (\nabla_\beta\phi_{,n})n_\alpha + \phi_{,n}(\nabla_\beta n_\alpha) + (\nabla_\beta\phi_{,a})e^{(a)}{}_\alpha + \phi_{,a}\nabla_\beta e^{(a)}{}_\alpha \tag{9}$$

and further expand

$$\nabla_\beta\phi_{,n} = \phi_{,nn}n_\beta + \phi_{,na}e^{(a)}{}_\beta \tag{10}$$

and

$$\nabla_\beta\phi_{,a} = \phi_{,an}n_\beta + \phi_{,ab}e^{(b)}{}_\beta. \tag{11}$$

Here

$$\begin{aligned}\phi_{,nn} &= (n^\alpha\phi_{,\alpha})_{,\beta}n^\beta = (\phi_{,n})_{,\beta}n^\beta\\ \phi_{,ab} &= (e_{(a)}{}^\alpha\phi_{,\alpha})_{,\beta}e_{(b)}{}^\beta = (\phi_{,a})_{,\beta}e_{(b)}{}^\beta\\ \phi_{,na} &= (\phi_{,n})_{,\beta}e_{(a)}{}^\beta\end{aligned} \tag{12}$$

and

$$\phi_{,an} = (\phi_{,a})_{,\beta} n^{\beta}. \tag{13}$$

Note that expressions (12) and (13) for $\phi_{,na}$ and $\phi_{,an}$ are not equivalent since second-order tetrad derivatives do not in general commute. These expressions do, however, separate naturally into a term that is symmetric in a and n, denoted by $\phi_{,(na)}$, and terms that are antisymmetric in a and n, denoted by $\phi_{,[an]}$ and $\bar{\phi}_{,[na]}$. We thus write the derivatives (12) and (13) in the form

$$\phi_{,an} = \phi_{,(na)} + \phi_{,[an]} \tag{14}$$

and

$$\phi_{,na} = \phi_{,(na)} + \bar{\phi}_{,[na]}$$

where, since $\phi_{,a}$ and $\phi_{,n}$ are scalars,

$$\begin{aligned} \phi_{,(na)} &= n^{\alpha} e_{(a)}{}^{\beta} (\nabla_{\beta} \phi_{,\alpha}) \\ \phi_{,[an]} &= (\nabla_{\beta} e_{(a)}{}^{\alpha}) n^{\beta} \phi_{,\alpha} \end{aligned} \tag{15}$$

and

$$\bar{\phi}_{,[na]} = (\nabla_{\beta} n^{\alpha}) e_{(a)}{}^{\beta} \phi_{,\alpha} = K_a{}^b \phi_{,b}. \tag{16}$$

Here $K_a{}^b$ is the extrinsic curvature three-tensor of Σ (Israel 1966).

Now consider the tetrad components of $\nabla_{\beta} n_{\alpha}$. It can be shown that

$$\nabla_{\beta} n_{\alpha} = J_{ab} e^{(a)}{}_{\alpha} e^{(b)}{}_{\beta} \tag{17}$$

where

$$J_{ab} = (\nabla_{\beta} n_{\alpha}) e_{(a)}{}^{\alpha} e_{(b)}{}^{\beta} \tag{18}$$

is the only non-vanishing tetrad component. We can project $\nabla_{\beta} e^{(a)}{}_{\alpha}$ in a similar manner to obtain

$$\nabla_{\beta} e^{(a)}{}_{\alpha} = L^a{}_b e^{(b)}{}_{\alpha} n_{\beta} - K^a{}_c n_{\alpha} e^{(c)}{}_{\beta} + \Gamma^a_{bc} e^{(b)}{}_{\alpha} e^{(c)}{}_{\beta} \tag{19}$$

where

$$L^a{}_b = (\nabla_{\beta} e^{(a)}{}_{\alpha}) e_{(b)}{}^{\alpha} n^{\beta}$$

and Γ^a_{bc} are the three-dimensional Christoffel symbols of Σ (Israel 1966).

Now substitute equations (10)–(12) and (15)–(19) into the expression (9) to obtain the tetrad components of $\nabla_{\beta} \phi_{,\alpha}$ in the form

$$\nabla_{\beta} \phi_{,\alpha} = \phi_{,nn} n_{\alpha} n_{\beta} + \phi_{,(na)} e^{(a)}{}_{\alpha} n_{\beta} + \phi_{,(na)} n_{\alpha} e^{(a)}{}_{\beta} + (K_{ab} \phi_{,n} + \phi_{,ab} + \Gamma^c_{ab} \phi_{,c}) e^{(a)}{}_{\alpha} e^{(b)}{}_{\beta}. \tag{20}$$

Next contract equation (20) with $g^{\alpha\beta}$ to obtain

$$\Box \phi = \phi_{,nn} + X \tag{21}$$

where

$$X = K \phi_{,n} + {}^{(3)}g^{ab} (\phi_{,ab} + \Gamma^c_{ab} \phi_{,c}). \tag{22}$$

The jump in the normal derivative of ϕ across Σ is now obtained by integrating equation (6) through a layer of finite thickness 2ε with the expression (21) used on the left-hand side and the right-hand side evaluated by the limit (19). The integral of X will vanish in the limit $\varepsilon \to 0$ because the continuity of ϕ across Σ guarantees that $\phi_{,n}$ does not contain a delta function. This leaves

$$[\phi_{,n}] = \frac{8\pi}{2\omega+3} S \tag{23}$$

as the junction condition for $\phi_{,n}$.

2.3. *Generalised Lanczos equations*

In general relativity the jumps across Σ in the components of the extrinsic curvature K_{ab} are given by the Lanczos equations

$$[K_{ab}] \equiv \gamma_{ab} = -8\pi(S_{ab} - \tfrac{1}{2}g_{ab}S) \tag{24}$$

(Israel 1966). We derive in this section the generalisation of equations (25) to the Brans–Dicke theory.

The field equations (4) and (5) can be written in the alternative form

$$R_{\alpha\beta} = -8\pi\phi^{-1}\left(T_{\alpha\beta} - \frac{\omega+1}{2\omega+3} g_{\alpha\beta}T\right) - \omega\phi^{-2}\phi_{,\alpha}\phi_{,\beta} - \phi^{-1}\nabla_\beta\phi_\alpha \tag{25}$$

and at this point it is most convenient to work in Gaussian-normal coordinates (Synge 1960) because they greatly simplify the calculations. In these coordinates the components of equations (25) that are tangent to Σ become

$$^{(4)}R_{ab} = {}^{(3)}R_{ab} + K_{ab,n} + KK_{ab} - 2K_a{}^cK_{cb}.$$

This expression can be combined with equations (8), (20) and (24) to produce

$$K_{ab,n} = -8\pi\phi^{-1}\left({}^{(4)}T_{ab} - \frac{\omega+1}{2\omega+3} g_{ab}{}^{(4)}T\right) - Y_{ab} \tag{26}$$

where

$$Y_{ab} = {}^{(3)}R_{ab} + KK_{ab} - 2K_a{}^cK_{cd} - \omega\phi^{-2}\phi_{,a}\phi_{,b} - \phi^{-1}(\phi_{,ab} + \Gamma^c_{ab}\phi_{,c} - K_{ab}\phi_{,n}).$$

We now integrate equation (26) through a layer of finite thickness 2ε and take the limit $\varepsilon \to 0$. Since the integral of Y_{ab} vanishes when $\varepsilon \to 0$, we are left with

$$\gamma_{ab} = -8\pi\phi^{-1}\left(S_{ab} - \frac{\omega+1}{2\omega+3} g_{ab}S\right) \tag{27}$$

which is the required generalisation of the Lanczos equations. In the limit $\omega \to \infty$ equations (27) reduce to equations (24) since in this limit $\phi^{-1} \to 1$. Although Gaussian-normal coordinates were used to derive equations (27), they are invariant under general coordinate transformations since they contain only tetrad components. An alternative form for these equations is

$$\gamma_{ab} - \frac{\omega+1}{\omega} g_{ab}\gamma = -8\pi\phi^{-1}S_{ab} \tag{28}$$

where

$$\gamma = {}^{(3)}g^{ab}\gamma_{ab}$$

so that

$$\gamma = \frac{8\pi\omega}{2\omega+3}\phi^{-1}S,$$

a relation that is required in the following section.

2.4. Junction conditions for the gravitational field

We begin by projecting the scalar field stress–energy tensor $\Phi_{\alpha\beta}$ (equation (5)) into directions normal and tangential to Σ to obtain

$$\Phi_{nn} = \tfrac{1}{2}\omega\phi^{-2}(\phi_{,n}^{2} - \phi^{,a}\phi_{,a}) + \phi^{-1}\phi_{,nn} \tag{29}$$

and

$$\Phi_{an} = \omega\phi^{-2}\phi_{,a}\phi_{,n} + \phi^{-1}\phi_{,(na)}. \tag{30}$$

Outside the hypersurface the Brans–Dicke field equations hold with $T_{\alpha\beta} = 0$, so that on each side of Σ

$$G^{\pm}_{\alpha\beta} = -\Phi^{\pm}_{\alpha\beta}.$$

The tetrad components of these equations, when combined with the Gauss–Codassi equations (Israel 1966), give

$$G^{\pm}_{an} = K^{\pm}_{,a} - K^{\pm}{}_{a}{}^{b}{}_{;b} = -\Phi^{\pm}_{an} \tag{31}$$

and

$$-2G^{\pm}_{nn} = {}^{(3)}R + K^{2}_{\pm} - K^{\pm}_{ab}K^{ab}_{\pm} = \Phi^{\pm}_{nn}. \tag{32}$$

Before we can obtain the junction conditions by adding and subtracting the pairs of equations (31) and (32), we need to calculate the jumps across Σ in the tetrad components Φ_{an} and Φ_{nn}. The evaluation of $[\Phi_{an}]$ entails the evaluation of $[\Phi_{,(na)}]$, a task that can be accomplished by using equation (23). The derivative with respect to ξ^{a} of this equation is

$$[\phi_{,na}] = \frac{8\pi}{2\omega+3}S_{,a}. \tag{33}$$

It then follows from equations (15) and (17) that

$$\phi_{,na} = \phi_{,(na)} - K_{a}{}^{b}\phi_{,b}$$

which can then be combined with equations (27) and (33) to give

$$[\phi_{,(na)}] = \frac{8\pi}{2\omega+3}S_{,a} + 8\pi\phi^{-1}\left(S_{a}{}^{b}\phi_{,n} - \frac{\omega+1}{2\omega+3}S\phi_{,a}\right). \tag{34}$$

Consequently, the jump in Φ_{an} is

$$[\Phi_{an}] = \frac{8\pi}{2\omega+3}(\phi^{-1}S)_{,a} + 8\pi\phi^{-2}\phi^{,b}S_{ab}. \tag{35}$$

We next consider the expression for $[\Phi_{nn}]$. This involves the derivation of $[\phi_{,nn}]$, a task that is not so straightforward as the derivation of (34) because it involves the second derivative with respect to n. In this case one is not at liberty to differentiate equation (23) with respect to n because the direction n has no meaning for the strictly three-dimensional quantity $S={}^{(3)}S(\xi^a)$. Instead, we must return to the four-dimensional wave equation (6) and differentiate it with respect to n, using the expressions (21) and (22) for the left-hand side. This gives

$$\phi_{,nnn}+(K\phi_{,n})_{,n}=\frac{8\pi}{2\omega+3}\,{}^{(4)}T_{,n}. \tag{36}$$

As before, we integrate equation (36) through a layer of finite thickness and let the thickness vanish to obtain

$$[\phi_{,nn}]+[K\phi_{,n}]=\frac{8\pi}{2\omega+3}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}{}^{(4)}T_{,n}\,\mathrm{d}n. \tag{37}$$

The integral in this equation vanishes in the limit $\varepsilon\to 0$ because ${}^{(4)}T\to\delta(n)\,{}^{(3)}S$ as $\varepsilon\to 0$, with the result that the integrand is proportional to the derivative of a delta function. Consequently, equation (37) reduces to

$$[\phi_{,nn}]=-\frac{8\pi\omega}{2\omega+3}\phi^{-1}S\tilde{\phi}_{,n}-\frac{8\pi}{2\omega+3}S\tilde{K} \tag{38}$$

which is the desired expression for $[\phi_{,nn}]$. Here $\tilde{A}$ denotes the mean value of the variable A across Σ. Note that the terms on the right-hand side of equation (38) would have been lost if we had simply differentiated equation (23) with respect to n. It now follows readily that

$$[\Phi_{nn}]=-\frac{8\pi}{2\omega+3}\phi^{-1}S\tilde{K}. \tag{39}$$

The mean values of Φ_{an} and Φ_{nn} follow in a similar manner from equations (29) and (30), and we obtain

$$\tilde{\Phi}_{an}=\omega\phi^{-2}\phi_{,a}\tilde{\phi}_{,n}+\phi^{-1}\tilde{\phi}_{,(na)} \tag{40}$$

and

$$\tilde{\Phi}_{nn}=\tfrac{1}{2}\omega\phi^{-2}\left(\tilde{\phi}_{,n}^{2}\omega+\frac{16\pi^2}{(2\omega+3)^2}S^2\right)-\tfrac{1}{2}\omega\phi^{-2}\phi^{,a}\phi_{,a}+\phi^{-1}\tilde{\phi}_{,nn}. \tag{41}$$

The conservation equations and the junction conditions for the gravitational field can now be obtained by combining equations (31) and (32) with equations (35) and (39)–(41). There results

$$S^{ab}{}_{;b}=0 \tag{42}$$

$$\tilde{K}_{,a}-\tilde{K}_a{}^b{}_{;b}=-\omega\phi^{-2}\phi_{,a}\tilde{\phi}_{,n}-\phi^{-1}\tilde{\phi}_{,(na)} \tag{43}$$

$$\begin{aligned}{}^{(3)}R+\tilde{K}^2-\tilde{K}_{ab}\tilde{K}^{ab}&=16\pi^2\phi^{-2}\left(S_{ab}S^{ab}-\frac{2\omega^2+3\omega+3}{(2\omega+3)^2}S^2\right)\\&\quad+\omega\phi^{-2}(\tilde{\phi}_{,n}^{2}-\phi^{,a}\phi_{,a})+2\phi^{-1}\tilde{\phi}_{,nn}\end{aligned} \tag{44}$$

and

$$\tilde{K}_{ab}S^{ab}=0. \tag{45}$$

We now discuss the meaning of the above junction conditions. Equation (42) expresses the energy–momentum balance in the shell and is identical to the equation obtained by Israel (1966) in general relativity, because in the Brans–Dicke theory there is no direct coupling between matter and the scalar field. The interaction between these two quantities is only an indirect one; the presence of matter *and* the scalar field combine to determine the geometry, which in turn acts upon the matter. The effect of the scalar field on the geometry can be seen explicitly in equations (43) and (44). Equation (43) shows how the scalar field affects the mean value of the extrinsic curvature $\tilde{K}_{ab}$ of the shell, while equation (44) shows the combined effect of the surface distribution of matter S_{ab} in the shell and the scalar field on the shell's extrinsic curvature and the intrinsic curvature ${}^{(3)}R$.

Equation (45) is identical to the general relativity result (Israel 1966) where the left-hand side represents the mean value of the normal force acting on the shell, i.e.

$$\widetilde{n_\alpha \nabla_\beta S^{\alpha\beta}} = -\tilde{K}_{ab} S^{ab} = 0.$$

In addition, the jump in the force is given by

$$[n_\alpha \nabla_\beta S^{\alpha\beta}] = 8\pi\phi^{-1}\left(S_{ab}S^{ab} \frac{\omega+1}{2\omega+3} S^2\right).$$

The tangential component of the force acting on the shell is

$$e^{(a)}{}_\alpha \nabla_\beta S^{\alpha\beta} = -S^{ab}{}_{;b}$$

which can be combined with the conservation equation (42) to give

$$[e^{(a)}{}_\alpha \nabla_\beta S^{\alpha\beta}] = 0$$

and

$$\widetilde{e^{(a)}{}_\alpha \nabla_\beta S^{\alpha\beta}} = 0.$$

These are identical to the general relativity results (Israel 1966).

3. Equations of motion

The equations of motion for singular hypersurfaces follow directly from the junction conditions (27), (42) and (45) once the intrinsic stress–energy tensor S_{ab} is specified. For simplicity we assume the hypersurface consists of a perfect fluid with

$$S_{ab} = (p+\sigma)U_a U_b + p g_{ab} \tag{46}$$

where p is the surface pressure, σ is the surface density of mass–energy and U_a is the three-velocity of the matter comprising the hypersurface. As a consequence of equation (46),

$$S = 2p - \sigma. \tag{47}$$

The equation of motion can be derived in a manner analogous to that used by Kuchař (1968) for charged shells in general relativity. Kuchař examined the four-accelerations of an element of fluid in the shell as observed from the surrounding regions $V^\pm$ of space–time and derived the following expressions for the tangential and normal acceleration components:

$$e_{(a)\alpha} \frac{DU^\alpha}{D\tau}\bigg|^{\pm} = -(p+\sigma)^{-1} p_{,b}(U_a U^b + \delta_a{}^b) \tag{48}$$

and

$$n_\alpha \frac{DU^\alpha}{D\tau}\bigg|^{\pm} = -U^a U^b K^{\pm}_{ab}. \tag{49}$$

Here τ denotes the proper time of ideal clocks moving with the element and U^α is the four-dimensional extension of U^a as defined by Israel (1966). Equations (48) and (49) hold in the Brans–Dicke theory as well as in general relativity because they follow directly from equation (42) and

$$\frac{DU^\alpha}{D\xi^b} = U^a{}_{;b}e_{(a)}{}^\alpha - U^a K_{ab} n^\alpha$$

(Israel 1966). As Kuchař points out, equation (48) describes for observers in $V^\pm$ how matter in the shell streams within the shell itself, and thus makes no reference to changes in the shape or size of the shell. It is the normal components (49) that refer to changes in shape and size, and these are the equations that are required to analyse the dynamics of hypersurfaces. We thus add and subtract the pairs of equations (49) and combine them with equations (27), (44) and (46) to obtain the equations of motion in the form

$$\left[n_\alpha \frac{DU^\alpha}{D\tau}\right] = 8\pi\phi^{-1}\left(2\frac{\omega+1}{2\omega+3}p + \frac{\omega+2}{2\omega+3}\sigma\right) \tag{50}$$

and

$$\widetilde{n_\alpha \frac{DU^\alpha}{D\tau}} = (p+\sigma)^{-1}\tilde{K}p. \tag{51}$$

Equation (51) is identical to the general relativity equation, and in the limit $\omega \to \infty$ equation (50) reduces to

$$\left[n_\alpha \frac{DU^\alpha}{D\tau}\right] = 8\pi(p + \tfrac{1}{2}\sigma)$$

as given by Kuchař (1968).

For a hypersurface which consists of incoherent dust, $p = 0$ and the above equations of motion reduce to

$$\left[n_\alpha \frac{DU^\alpha}{D\tau}\right] = \frac{8\pi(\omega+2)}{2\omega+3}\phi^{-1}\sigma \tag{52}$$

and

$$\widetilde{n_\alpha \frac{DU^\alpha}{D\tau}} = 0. \tag{53}$$

In general relativity equations (52) and (53) were written out explicitly and solved analytically for a spherical shell of dust by Israel (1966, 1967b). The absence of gravitational radiation during spherical collapse makes the study of the collapse and dynamics of spherical shells a relatively simple matter in general relativity, a situation which does not extend to the Brans–Dicke theory because of the presence of the scalar field. In this case equations (52) and (53) must be solved in conjunction with the four-dimensional wave equation

$$\Box\phi = 0 \tag{54}$$

and the field equations

$$G_{\alpha\beta} = -\Phi_{\alpha\beta} \tag{55}$$

which hold in $V^{\pm}$. Although a spherical shell in general relativity moves in a Schwarzschild background (with space–time Minkowskian in the interior of the shell), the background geometry through which a spherical shell moves in the Brans–Dicke theory is an unknown dynamic entity. This must be determined from equations (54) and (55), and the necessity to do so considerably complicates the analysis. In spite of this, the analysis of a collapsing shell is still simpler to handle than the collapse of a neutron star, and there is a good chance that the fate of the scalar field during the collapse will be able to be followed numerically when this problem is analysed in detail.

We reserve a discussion of this problem for a later publication, but as a simple application of the formulae presented in § 2, the following section considers the structure of a *static* spherical shell.

4. Static spherical shell

The simplest application of the formulae presented in § 2 is to calculate the structure of a static spherical shell of perfect fluid. We take $V^{\pm}$ to be the exterior of the shell Σ with coordinates $x_\alpha^{+} = (t, r, \theta, \varphi)$; V^{-} is then the interior of the shell (the four-dimensional space–time enclosed by the shell) with coordinates $x_\alpha^{-} = (T, r, \theta, \varphi)$, and as intrinsic coordinates we take $\xi^a = (\tau, \theta, \varphi)$.

In V^{+} the line element in isotropic form is

$$ds^2 = -e^{2\nu}\, dt^2 + e^{2\beta}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) \tag{56}$$

and the appropriate solution of the field equations (56) is the Brans type I metric

$$e^{\nu} = e^{\nu_0}\left(\frac{1 - B/r}{1 + B/r}\right)^{-1/\lambda} \tag{57}$$

and

$$e^{\beta} = e^{\beta_0}\left(1 + \frac{B}{r}\right)^2 \left(\frac{1 - B/r}{1 + B/r}\right)^{(\lambda - A - 1)\lambda} \tag{58}$$

(Brans and Dicke 1961, Brans 1962). Here

$$\lambda = [(A+1)^2 - A(1 - \tfrac{1}{2}\omega A)]^{1/2} \tag{59}$$

and ν_0, β_0, B and A are constants. Asymptotic flatness requires $\nu_0 = \beta_0 = 0$ and

$$B = M\lambda/2 \tag{60}$$

where M is the total mass–energy of the shell (Salmona 1967). Of the constants that appear in expressions (57) and (58) it is only A that remains to be determined.

In the interior of Σ space–time is Minkowskian, as it is in general relativity, because a singularity exists at the origin unless the scalar field ϕ is constant in the interior (Lightman *et al* 1975, problem 16.5). When the scalar field is constant, the field equations reduce to those of general relativity, for which the only solution regular at the origin is flat space–time. The interior line element is thus taken to be

$$ds^2 = -dT^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

For the coordinate radius of the shell we take $r = R_-$ as measured in V^- and $r = R_+$ as measured in V^+. It is not necessary that $R_- = R_+$ or that $T = t$, since the coordinate patches in $V^\pm$ need not join smoothly across Σ.

The triad of vectors $e^{\pm}_{(a)}{}^{\alpha}$ tangent to the shell are

$$e^{\pm}_{(\tau)}{}^{\alpha} = [(-g^{\pm}_{00})^{-1/2}, 0, 0, 0]$$

$$e^{\pm}_{(\theta)}{}^{\alpha} = (0, 0, 1, 0)$$

and

$$e^{\pm}_{(\varphi)}{}^{\alpha} = (0, 0, 0, 1)$$

and in addition the unit normal vector is

$$n^{\pm} = (0, g_{rr}{}^{-1/2}, 0, 0).$$

The intrinsic metric induced in Σ is

$$ds^2 = -d\tau^2 + R_+{}^2 e^{2\beta_+}(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

where

$$e^{\beta_+} = \left(1 + \frac{B}{R_+}\right)^2 \left(\frac{1 - B/R_+}{1 + B/R_+}\right)^{(\lambda - A - 1)/\lambda}.$$

Since the two four-dimensional metrics $g^{\pm}_{\alpha\beta}$ both induce the same intrinsic metric in Σ, the coordinate radii R_- and R_+ are related by

$$R_- = R_+ e^{\beta_+}.$$

In V^+ the solution to equation (54) for the scalar field is

$$\phi^{(\text{ext})} = \frac{2\omega + 4}{2\omega + 3} \left(\frac{1 - B/r}{1 + B/r}\right)^{A/\lambda} \tag{61}$$

(Brans and Dicke 1961, Brans 1962). The constant value ϕ_0 of the scalar field in V^- then follows from the junction condition (7) as

$$\phi_0 = \frac{2\omega + 4}{2\omega + 3} \left(\frac{1 - B/R_+}{1 + B/R_+}\right)^{A/\lambda}. \tag{62}$$

The jump in the normal derivative of ϕ across Σ which is given by the junction condition (23) can be used to obtain the following relation between the parameters of the shell and S:

$$\frac{AB(\omega + 2)R_+{}^2}{2\pi\lambda} = S(R_+ + B)^{(2\lambda + 2A + 1)/\lambda}(R_+ - B)^{(2\lambda - 2A - 1)/\lambda}. \tag{63}$$

We now calculate the jump in the components of the extrinsic curvature K_{ab}, and the results are

$$\gamma_{\tau\tau} = -\frac{2BR_+{}^2}{\lambda}(R_+ + B)^{-(2\lambda + A + 1)/\lambda}(R_+ - B)^{-(2\lambda - A - 1)/\lambda} \tag{64}$$

$$\gamma_{\theta\theta} = -\frac{2B}{R_+}\left(\frac{A + 1}{\lambda} R_+ - B\right)(R_+ + B)^{(A + 1)/\lambda}(R_+ - B)^{-(A + 1)/\lambda} \tag{65}$$

and

$$\gamma_{\varphi\varphi} = \sin^2\theta\, \gamma_{\theta\theta}.$$

All other components are zero, and in addition

$$\gamma = -2BR_+\left(\frac{2A+1}{\lambda}R_+ - 2B\right)(R_+ + B)^{-(2\lambda+A+1)/\lambda}(R_+ - B)^{-(2\lambda-A-2)/\lambda}.$$

It now follows from the Lanczos equations in the form (28) that the non-vanishing components of the surface stress–energy tensor can be expressed in terms of the parameters of the shell as

$$S_{\tau\tau} = \frac{(\omega+2)BR_+{}^2}{2\pi\omega(2\omega+3)\lambda}[2A(\omega+1)+2\omega+1+2(\omega+1)B\lambda]$$
$$\times (R_+ + B)^{-(2\lambda+2A+1)/\lambda}(R_+ - B)^{-(2\lambda-2A-1)/\lambda}$$

$$S_{\theta\theta} = -\frac{(\omega+2)B}{2\pi\omega(2\omega+3)\lambda R_+}[(A\omega+2A+1)R_+ + (\omega+2)B\lambda](R_+ + B)^{1/\lambda}(R_+ - B)^{-1/\lambda}$$

and

$$S_{\varphi\varphi} = \sin^2\theta\, S_{\theta\theta}.$$

The discussion so far has been general in that no mention has been made of the nature of the material comprising the shell. In order to make further progress this must be specified and for simplicity we assume again that the matter is a perfect fluid with stress–energy tensor (46). For a static spherical shell U_a has components $U_a = (1, 0, 0)$ so that the components of S_{ab} are

$$S_{\tau\tau} = \sigma$$

and

$$S_{\theta\theta} = R_+{}^{-2}(R_+ + B)^{2(\lambda+A+1)/\lambda}(R_+ - B)^{2(\lambda-A-1)/\lambda}p.$$

The simplest way to derive expressions for p and σ is to use equations (47) and (63) in combination with equation (64) and the $\tau\tau$ component of equations (28):

$$\gamma_{\tau\tau} = -8\pi\phi^{-1}\left(2\frac{\omega+1}{2\omega+3}p + \frac{\omega+2}{2\omega+3}\sigma\right).$$

The results are

$$p = \frac{B(\omega A+2A+1)(\omega+2)R_+{}^2}{4\pi(2\omega+1)\lambda}(R_+ + B)^{-(2\lambda+2A+1)/\lambda}(R_+ - B)^{-(2\lambda-2A-1)/\lambda} \qquad (66)$$

$$\sigma = \frac{B(1-\omega A-A)(\omega+2)R}{2\pi(2\omega+3)\lambda}(R_+ + B)^{-(2\lambda+2A+1)/\lambda}(R_+ - B)^{-(2\lambda-2A-1)/\lambda}. \qquad (67)$$

The solution of the Brans–Dicke field equations that represents a static spherical shell of perfect fluid with a specified total mass–energy M and exterior coordinate radius R_+ will now be completely determined once the remaining unknown constant A is found. A follows most readily by combining expressions (65)–(67) with the $\theta\theta$ component of equations (27):

$$\gamma_{\theta\theta} = -8\pi\phi^{-1}R_+{}^2e^{2\beta_+}\left(\frac{p}{2\omega+3} + \frac{\omega+1}{2\omega+3}\sigma\right).$$

It is found that A satisfies the quadratic equation

$$\alpha(\omega+2)A^2-(\omega+2-2\alpha)A+2\alpha-1=0$$

where $\alpha=M/2R_+$. The appropriate root here for A is

$$A=-\frac{1}{\omega+2}+\frac{1}{2\alpha}\left[1-\left(1-4\alpha^2\frac{2\omega+3}{(\omega+2)^2}\right)^{1/2}\right] \tag{68}$$

because this expression reduces to the weak-field limit

$$A=-\frac{1}{\omega+2}$$

when $\alpha \ll 1$ (Brans and Dicke 1961). Note that expressions (58)–(63) and (67)–(69), which are the complete solution for the static spherical shell, depend only on the single parameter α.

We now discuss some of the physical properties of this solution with particular emphasis on the limits that must be placed on α. One limit is obtained from expression (68) for A, which is only real when

$$\alpha \leqslant \frac{\omega+2}{2(2\omega+3)^{1/2}} \equiv \alpha_1(\omega). \tag{69}$$

It is also necessary for the quantity $1-B/r$ which appears in the exterior metric to be positive for all values of $R \geqslant R_+$, a requirement that entails

$$\alpha < \left(\frac{2\omega}{2\omega+3}\right)^{1/2} \equiv \alpha_2(\omega). \tag{70}$$

In the limit $\alpha=\alpha_2$, g_{00} vanishes at the shell $r=R_+$ because, when $\alpha=\alpha_2$, $B=R_+$. Consequently, in a sequence of *static* configurations an event horizon forms at $r=R_+$ when $\alpha=\alpha_2$ since the surface $g_{00}=0$ is an event horizon in static spherical geometries (Israel 1967c). As the scalar field is not constant outside the horizon, the solution represented here is not the Schwarzschild solution and consequently the event horizon is singular. This is a result of the fact that the Schwarzschild solution is the only static spherical vacuum solution of the Brans–Dicke field equations that possesses a non-singular event horizon (Johnson 1972). The solution with $\alpha=\alpha_2$ thus represents a naked singularity.

It is a simple matter to verify that the event horizon is singular by calculating the physical components $R^{(\alpha)}{}_{(\beta)(\gamma)(\delta)}$ of the Riemann tensor at $r=R_+$. If any of these are singular, the event horizon is singular. For the metric (56) a typical physical component is

$$R^{(0)}{}_{(2)(0)(2)}=-\frac{2B}{\gamma}[(r-B)^2+2BC]r^3(r+B)^{2(C-3)}(r-B)^{-2(C+1)} \tag{71}$$

where $C=(\lambda-A-1)/\lambda$. When $\alpha=\alpha_2$, $C+1$ can be written as

$$C+1=\frac{\omega+1}{\omega+2}\left[2-\left(\frac{2\omega}{2\omega+3}\right)^{1/2}\right]$$

which satisfies $C+1>0$ for all values of ω in the range $-1<\omega<\infty$. The exponent of $r-B$ in expression (71) is thus negative, with the result that the horizon is singular. All other non-zero physical components $R^{(\alpha)}{}_{(\beta)(\gamma)(\delta)}$ are also singular at $r=R_+$ except in the general relativity limit $\omega \to \infty$, in which case all physical components are finite everywhere except at $r = 0$.

A further limit on α can be obtained by considering the relation between p and σ. It follows from equations (67) and (68) that p and σ are related by

$$p = \frac{1}{2} \frac{1 + (\omega + 2)A}{1 - (\omega + 1)A} \sigma$$

which may be regarded as an equation of state. In the limit $\omega \to \infty$ this expression reduces to the general relativity result

$$p = \frac{\alpha}{2(1-\alpha)} \sigma$$

and in the weak-field limit $\alpha \ll 1$ it follows, as expected, that $p \ll \sigma$. The speed at which small disturbances propagate over the shell is

$$v_s = \left(\frac{dp}{d\sigma}\right)^{1/2} = \left[\frac{1}{2} \frac{1 + (\omega + 2)A}{1 - (\omega + 1)A}\right]^{1/2}$$

which must not exceed unity. From this expression, $v_s \leq 1$ when α satisfies

$$\alpha \leq \frac{2(3\omega + 4)}{9\omega + 4} \equiv \alpha_s(\omega) \tag{72}$$

which coincides with the dominant energy condition $\sigma > p$. In the general relativity limit the inequality (72) becomes $\alpha < 2/3$ or $R_+ > 3M/4$, in agreement with Lightman *et al* (1975, problem 16.15).

The final quantity we consider is the redshift of a photon emitted from the surface of the shell and received at infinity. This is given by

$$z_s = \left(\frac{1 + \alpha\lambda}{1 - \alpha\lambda}\right)^{1/\lambda} - 1$$

and has the general relativity limit $z_s = 2\alpha/(1-\alpha)$.

To see which of the above limits on α is the strongest for a given value of the Dicke coupling constant ω, expressions (69), (70) and (72) are plotted as functions of ω in figure 1. For $\omega > 13/4$, $\alpha_3(\omega) < \alpha_2(\omega)$ holds so that $\alpha \leq \alpha_3$ is the strongest condition. That is, when $\omega > 13/4$ a sequence of static shells with increasing α becomes acausal before the naked singularity configuration is reached. This is, of course, the situation for the experimentally allowed range of ω. When $\omega < 13/4$ a naked singularity is reached before v_s exceeds unity. It is only for the value $\omega = 2$ that the limit $\alpha \leq \alpha_1(\omega)$ is relevant, because $\alpha_1(\omega) > \alpha_2(\omega)$ holds for all values of ω except $\omega = 2$, at which point $\alpha_1 = \alpha_2$.

Also plotted in figure 1 is the surface redshift z_s evaluated for shells with $\alpha = \alpha_3(\omega)$. This redshift is infinite when $\omega = 13/4$ (the naked singularity configuration) and decreases monotonically with increasing ω to the general relativity limit $z_s = 4$ when $\omega \to \infty$.

5. Summary and conclusions

We have derived here the junction conditions and equations of motion in an invariant form for uncharged singular time-like hypersurfaces in the Brans–Dicke theory of

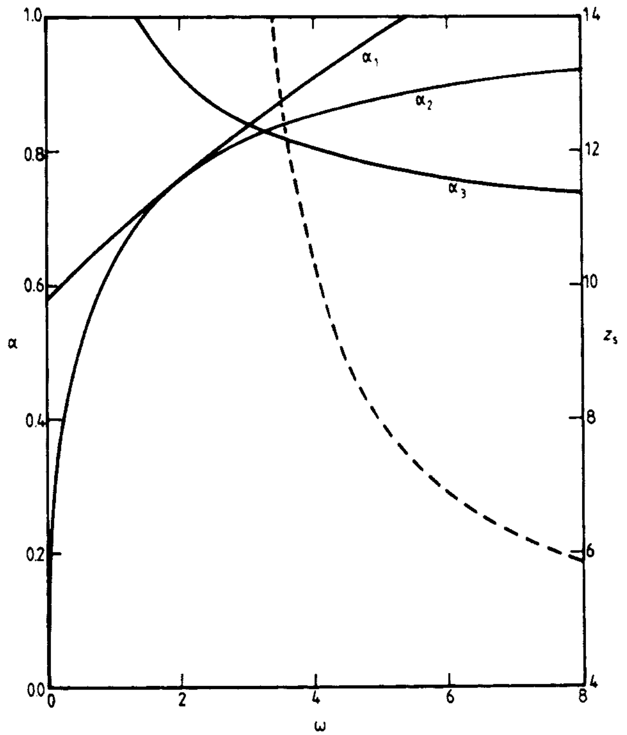

Figure 1. The shell parameter $\alpha = M/2R_+$ is constrained by the three limits $\alpha \leqslant \alpha_1(\omega)$, $\alpha < \alpha_2(\omega)$ and $\alpha \leqslant \alpha_3(\omega)$ where α_1, α_2 and α_3 are defined by equations (69), (70) and (72) respectively. See the text for an explanation of these limits. The figure displays α_1, α_2 and α_3 as functions of the Dicke coupling constant ω, and also shows the redshift z_s of a photon emitted from the surface of the shell and received at infinity, for shells with $\alpha = \alpha_3$.

gravity. The derivation was based on the methods developed by Israel (1966) for singular uncharged hypersurfaces in general relativity, and by Kuchař (1968) for charged hypersurfaces. The junction conditions—equations (7), (23), (27) and (42)–(45)—describe how the scalar field and the gravitational field change across the hypersurface, and the equations of motion—equations (50) and (51)—describe how the hypersurface moves through space–time. Since there is no direct coupling between matter and the scalar field in the Brans–Dicke theory, some of the junction conditions and some of the equations of motion have forms identical to those in general relativity.

In § 4 we presented an exact solution of the Brans–Dicke field equations which represents a static spherical shell of perfect fluid. Space–time is flat inside the shell and all the constants which appear in the Brans type I solution for the gravitational field outside the shell are expressed in terms of the single parameter $\alpha = M/2R_+$. Here M is the total mass–energy of the shell and R_+ is the coordinate radius (in isotropic coordinates), as measured from outside the shell.

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